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A computational solution of hydromagnetic-free convective flow past a vertical plate in a rotating heat-generating fluid with Hall and ion-slip currents

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SUMMARY

Taking Hall and ion-slip current into account, the unsteady magnetohydrodynamic heat-generating free convective flow of a partially ionized gas past an infinite vertical plate in a rotating frame of reference is investigated theoretically. A computer program using finite elements is employed to solve the coupled non-linear differential equations for velocity and temperature fields. The effects of Hall and ion-slip currents as well as the other parameters entering into the problem are discussed extensively and shown graphically.

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KEY WORDS: finite element method; magnetohydrodynamic free convective flow; rotating heat-generating fluid; computational solution

1. INTRODUCTION

The study of fluid flows has important applications in engineering. For example, the effect of an applied magnetic field on unsteady free convection flow along a vertical plate has been given considerable attention because of its application in the cooling of nuclear reactors or in the study of the structures of stars and planets. In recent years, considerable interest has been given to the theory of rotating fluids due to its application in cosmic and geophysical sciences. In an ionized gas where the density is low and/or the magnetic field is very strong, the effects of Hall and ion-slip currents play a significant role in the velocity distribution of the flow. The study of magnetohydrodynamic flows with Hall and ion-slip currents has important engineering applications in the problems of magnetohydrodynamic generators and of Hall accelerators as well as flight magnetohydrodynamics. The problem of viscous flows with Hall current has been solved by a number of authors [1–4].

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Naroua *et al*. [5] looked at the MHD Stokes problem of convective flow from a vertical infinite plate in a rotating fluid where they proposed a new mathematical formulation. Cookey *et al*. [6] proposed a model for the study of MHD free-convection flow past an infinite heated vertical plate in a porous medium in which they observed that increased cooling of the plate was accompanied with an increase in the velocity. Taneja and Jain [7] looked at the unsteady MHD flow in a porous medium in the presence of radiative heat where they obtained expressions for the velocity, temperature and rate of heat transfer. Ghosh [8] attempted to study the MHD flow of a dusty, electrically conducting, visco-elastic Rivlin–Ericksen fluid starting from the rest with time-dependent types applied at the free surface. He solved the problem analytically by using the Laplace transform technique. Debnath [9–11] has made a good contribution to the unsteady hydrodynamic and hydromagnetic boundary layer flows with or without Hall current effects in a rotating fluid system. When the conducting fluid is a partially ionized gas, e.g. water gas seeded with potassium, the Hall and ion-slip currents are also significant. Hence, the aim of the present work is to study the unsteady free convective flow of a partially ionized gas past an infinite vertical plate. Here, (i) the plate is subjected to a constant suction velocity, (ii) the fluid and the plate are in a state of rigid rotation with a uniform angular velocity Ω about the z'-axis, (iii) a strong magnetic field of uniform strength is applied transversely to the direction of the flow and (iv) the heat source Q^* is of the type $Q^* = Q(T'_\infty - T')$. A computer program using finite elements is employed to solve the coupled non-linear differential equations for velocity and temperature fields.

2. MATHEMATICAL ANALYSIS

Consider the unsteady MHD free-convective flow of a partially ionized fluid past an infinite vertical plate when the fluid and the plate rotate in unison with a constant angular velocity Ω about z'-axis taken normal to the plate as shown in Figure 1.

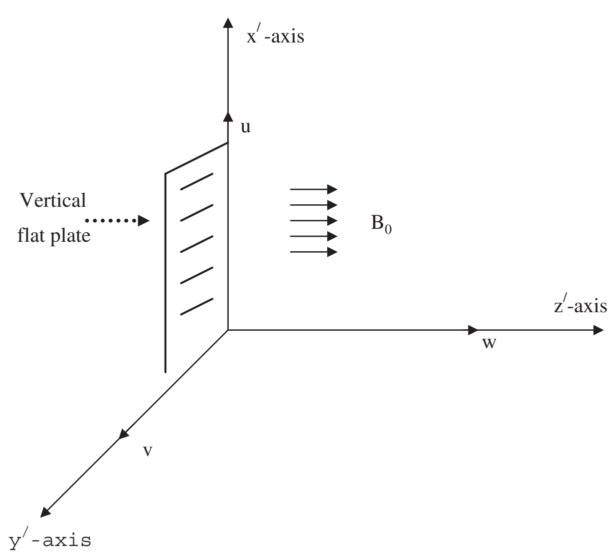

Figure 1. The flow configuration with the coordinate system.

A uniform strong magnetic field B_0 is imposed along the z'-axis and the plate is taken as electrically non-conducting. Since the plate is infinite in extent, all the physical variables except pressure depend on z' and t' only. The equation of continuity $\nabla \cdot \mathbf{q} = 0$ gives $w' = -w_0$ (>0), where $\mathbf{q} = (u', v', w')$. The flow has a low Mach number, and hence the density of the ionized gas can be taken as constant. However, for such a fluid, the Hall and ion-slip currents significantly affect the flow in the presence of large magnetic fields. The induced magnetic field is neglected, since the magnetic Reynolds number of a partially ionized gas is very small [12]. The equation of conservation of electric change $\nabla \cdot \mathbf{J} = 0$ gives $J_{z'} = \text{constant}$, where $\mathbf{J} = (j_{x'}, j_{y'}, j_{z'})$. This constant is zero at the plate, which is electrically non-conducting. Thus, $j_{z'} = 0$ everywhere in the flow. When the strength of the magnetic field is very large, Ohm's law must be modified to include Hall and ion-slip currents [13]:

$$\mathbf{J} + \frac{\omega_e \tau_e}{B_0} \mathbf{J} \times \mathbf{B} = \sigma \left(\mathbf{E} + \mu_e \mathbf{q} \times \mathbf{B} + \frac{1}{e\eta_e} \nabla \cdot P_e \right) \tag{1}$$

where all the physical variables are given in nomenclature. Under the assumption that the electron pressure (for weakly ionized gas) and the thermoelectric pressure are negligible, Equation (1) becomes

$$\begin{aligned} j_{x'} &= [\alpha'(E_{x'} + B_0 v') - \beta'(E_{y'} + (U' - u')B_0)]\sigma \\ j_{y'} &= [\alpha'(E_{y'} + B_0(U' - u') + \beta'(E_{x'} + B_0 v'))]\sigma \end{aligned} \tag{2}$$

where

$$\begin{aligned} \alpha' &= \frac{(1 + \beta_e \beta_i)}{(1 + \beta_e \beta_i)^2 + \beta_e^2} \\ \beta' &= \frac{\beta_e}{(1 + \beta_e \beta_i)^2 + \beta_e^2}, \quad \beta_e = \omega_e \tau_e, \quad \beta_i = \omega_i \tau_i \\ \mathbf{E} &= (E_{x'}, E_{y'}, E_{z'}) \end{aligned} \tag{3}$$

In a rotating frame of reference, the equations of motion and energy are given by

$$\frac{\partial u'}{\partial t'} - w_0 \frac{\partial u'}{\partial z'} - 2\Omega v' = \frac{\partial u'}{\partial t'} + v \frac{\partial^2 u'}{\partial z'^2} + g\beta(T' - T'_\infty) + j_{y'} \frac{B_0}{\rho} \tag{4}$$

$$\frac{\partial v'}{\partial t'} - w_0 \frac{\partial v'}{\partial z'} + 2\Omega(u' - U') = v \frac{\partial^2 v'}{\partial z'^2} - \frac{1}{\rho} j_{x'} B_0 \tag{5}$$

$$\frac{\partial T'}{\partial t'} - w_0 \frac{\partial T'}{\partial z'} = \frac{k}{\rho C_p} \frac{\partial^2 T'}{\partial z'^2} + Q(T'_\infty - T) \tag{6}$$

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The boundary conditions are given by

$$
\begin{aligned}
&u' = 0, \quad v' = 0, \quad T' = T_w \quad \text{at } z' = 0 \\
&u' = U(t), \quad v' = 0, \quad T' = T_\infty \quad \text{as } z' \to \infty
\end{aligned}
\tag{7}
$$

Introducing the non-dimensional quantities

$$
\begin{aligned}
&z = w_0 \frac{z'}{v}, \quad u = \frac{u'}{U_0}, \quad v = \frac{v'}{U_0}, \quad t = \frac{t' w_0^2}{v}, \quad U = \frac{U'}{U_0} \\
&w = v \frac{w'}{w_0^2}, \quad \theta = \frac{(T' - T'_\infty)}{(T'_w - T'_\infty)}, \quad Pr = \rho v \frac{C_\mathrm{p}}{K} \\
&\mathrm{Er} = \frac{\Omega v}{w_0^2}, \quad Gr = v g \beta \frac{(T'_w - T'_\infty)}{U_0 w_0^2}, \quad M^2 = \frac{\sigma B_0^2 v}{\rho w_0^2}, \quad \delta = \frac{Q v^2}{k w_0^2} \\
&\mathbf{E} = E_x + \mathrm{i} E_y, \quad M_1 = 2\,\mathrm{Er} \cdot \mathrm{i} + B_0^2 \\
&E_x = \frac{E_{x'}}{B_0 w_0}, \quad E_y = \frac{E_{y'}}{B_0 w_0}, \quad j_x = \frac{j_{x'}}{\sigma B_0 w_0}, \quad j_y = \frac{j_{y'}}{\sigma B_0 w_0} \\
&\alpha_0 = \alpha' + \mathrm{i}\beta', \quad B_0^2 = M^2 \alpha_0
\end{aligned}
\tag{8}
$$

and using Equation (2) in Equations (4)–(6), we have

$$
\frac{\partial q}{\partial t} - \frac{\partial q}{\partial z} + M_1 (q - U) = \frac{\partial U}{\partial t} + \frac{\partial^2 q}{\partial z^2} + Gr\, \theta - \frac{M^2}{U_0} \mathrm{i} w_0 \alpha_0 \mathbf{E} \tag{9}
$$

Neglecting the electric field $\mathbf{E} = 0$, we have

$$
\frac{\partial q}{\partial t} - \frac{\partial q}{\partial z} + M_1 (q - U) = \frac{\partial U}{\partial t} + \frac{\partial^2 q}{\partial z^2} + Gr\, \theta \tag{10}
$$

$$
Pr \frac{\partial \theta}{\partial t} - Pr \frac{\partial \theta}{\partial z} = \frac{\partial_2 \theta}{\partial z^2} - \delta \theta \tag{11}
$$

where $q = u + \mathrm{i}v$.

The corresponding boundary conditions are given by

$$
\begin{aligned}
&q = 0, \quad \theta = 1 \quad \text{at } z = 0 \\
&q \to U(t) = 1 + \varepsilon\, \mathrm{e}^{\mathrm{i} w t}, \quad \theta \to 0 \quad \text{as } z \to \infty
\end{aligned}
\tag{12}
$$

where $\varepsilon \ll 1$.

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The above system of Equations (10) and (11) with boundary conditions (12) has been solved numerically by a computer program using the finite element method in Steps 1 and 2.

Step 1: We solve Equation (11) with the help of boundary conditions (12). Constructing the quasi-variational statement of Equation (11), we have

$$0 = \int_{\Omega} \phi_i \left\{ Pr\frac{\partial\theta}{\partial t} - Pr\frac{\partial\theta}{\partial z} - \frac{\partial_2\theta}{\partial z^2} + \delta\theta \right\} \mathrm{d}z \tag{13}$$

where ϕ_i is the test function and Ω is the region of the flow.

Consider an N-element mesh and a two-parameter (semi-discrete) Galerkin approximation [14] of the form

$$\theta(z,t) = \sum_{j=1}^{5} C_j(t)\phi_j(z) \tag{14}$$

with

$$\begin{aligned} \phi_1^{(i)}(z) &= \frac{(z_{i+1}-z)}{(z_{i+1}-z_i)} \\ \phi_2^{(i)}(z) &= \frac{(z-z_i)}{(z_{i+1}-z_i)} \end{aligned} \tag{15}$$

where $i = 1, 2, 3, \ldots, N$ and z_i and z_{i+1} are, respectively, the lower and upper coordinates of the element i.

Using Equations (14) and (15), Equation (13) reduces to

$$[A]\left\{\frac{\partial C}{\partial t}\right\} + [B]\{C\} = \{P\} \tag{16}$$

where

$$\begin{aligned} A_{ij} &= \int_{\Omega} Pr\phi_i\phi_j \,\mathrm{d}z \\ B_{ij} &= \int_{\Omega} \left(-Pr\phi_i\frac{\partial\phi_j}{\partial z} + \frac{\partial\phi_i}{\partial z}\frac{\partial\phi_j}{\partial z} - \delta\phi_i\phi_j \right) \mathrm{d}z \\ P_i &= 0 \end{aligned} \tag{17}$$

Using the Θ-family of approximation developed by Reddy [14], Equation (16) reduces to

$$[\hat{A}]\{C\}_{n+1} = [\hat{B}]\{C\}_n + \{\hat{P}\} \tag{18}$$

where

$$\begin{aligned} [\hat{A}] &= [A] + \Theta\Delta t[B] \\ [\hat{B}] &= [A] - (1-\Theta)\Delta t[B] \\ [\hat{P}] &= \Delta t[\Theta\{P\}_{n+1} + (1-\Theta)\{P\}_n] \end{aligned} \tag{19}$$

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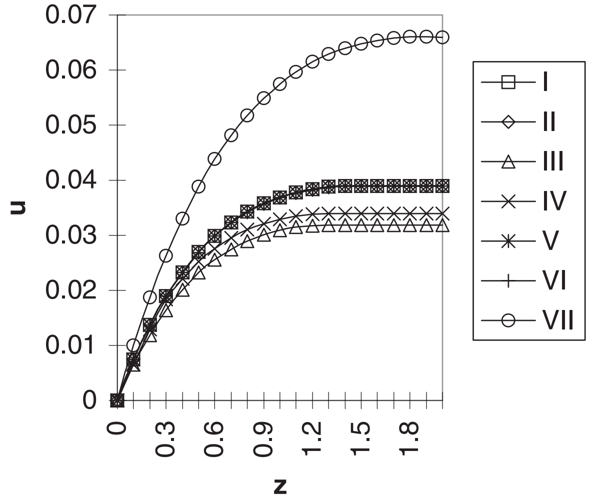

Series	Pr	β_e	β_i	Er	δ	t
I	0.71	1.00	0.00	0.01	0.10	0.001
II	7.00	1.00	0.00	0.01	0.10	0.001
III	0.71	1.50	0.00	0.01	0.10	0.001
IV	0.71	1.00	0.40	0.01	0.10	0.001
V	0.71	1.00	0.00	0.05	0.10	0.001
VI	0.71	1.00	0.00	0.01	0.50	0.001
VII	0.71	1.00	0.00	0.01	0.10	0.002

Figure 2. Transient primary velocity profiles (u) at $Gr=10$.

The initial value C_0 is obtained by the Galerkin method from a 32-element mesh and is given by

$$C_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{20}$$

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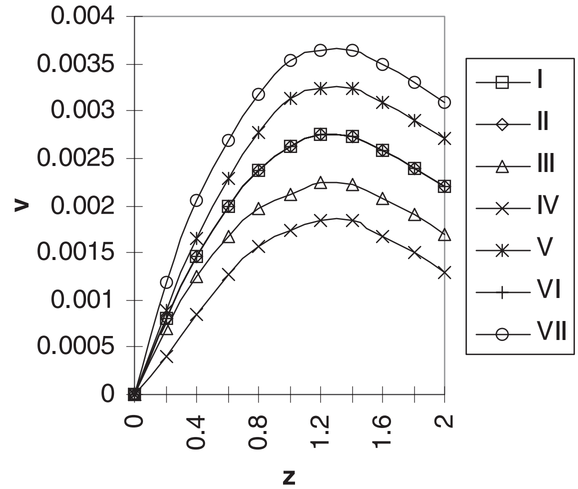

Series	Pr	β_e	β_i	Er	δ	t
I	0.71	1.00	0.00	0.01	0.10	0.001
II	7.00	1.00	0.00	0.01	0.10	0.001
III	0.71	1.50	0.00	0.01	0.10	0.001
IV	0.71	1.00	0.40	0.01	0.10	0.001
V	0.71	1.00	0.00	0.05	0.10	0.001
VI	0.71	1.00	0.00	0.01	0.50	0.001
VII	0.71	1.00	0.00	0.01	0.10	0.002

Figure 3. Transient secondary velocity profiles (v) at $Gr = 10$.

For $t > 0$,

$$\{C\}_{n+1} = [\hat{A}^{-1}][B]\{C\}_n + [\hat{A}^{-1}]\{\hat{P}\} \tag{21}$$

Step 2: We solve Equation (10) with the help of boundary conditions (12). Constructing the quasi-variational statement of Equation (10), we have

$$0 = \int_\Omega \Psi_i \left\{ \frac{\partial q}{\partial t} - \frac{\partial q}{\partial z} + M_1(q-U) - \frac{\partial_2 q}{\partial z^2} - Gr\,\theta - \frac{\partial U}{\partial t} \right\} \mathrm{d}z \tag{22}$$

where Ψ_i is the test function and Ω is the region of the flow.

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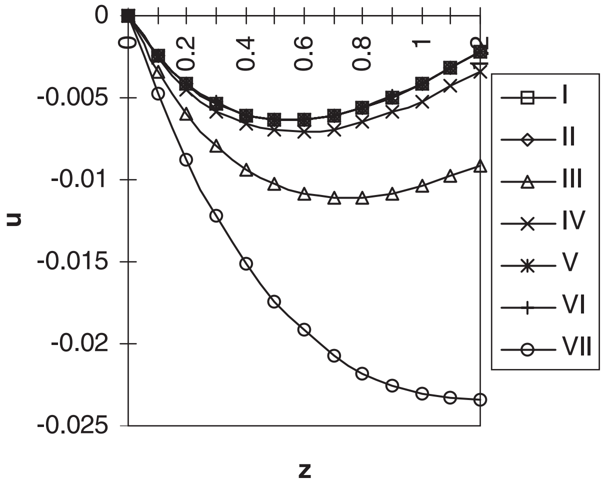


Series	Pr	β_e	β_i	Er	δ	t
I	0.71	1.00	0.00	0.01	0.10	0.001
II	7.00	1.00	0.00	0.01	0.10	0.001
III	0.71	1.50	0.00	0.01	0.10	0.001
IV	0.71	1.00	0.40	0.01	0.10	0.001
V	0.71	1.00	0.00	0.05	0.10	0.001
VI	0.71	1.00	0.00	0.01	0.50	0.001
VII	0.71	1.00	0.00	0.01	0.10	0.002

Figure 4. Transient primary velocity profiles (u) at $Gr=-10$.

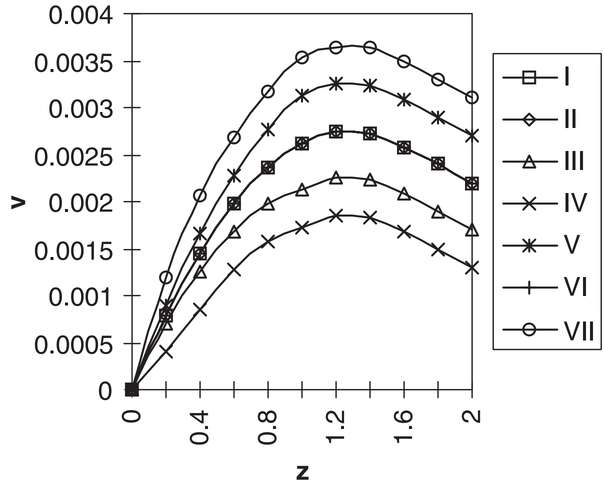
Consider a two-parameter (semi-discrete) Galerkin approximation of the form

$$q(z,t)=\sum_{j=1}^{N} \mathrm{d}_j(t)\Psi_j(z) \tag{23}$$

with

$$\begin{aligned}\Psi_1^{(i)}(z)&=\frac{(z_{i+1}-z)}{(z_{i+1}-z_i)}(1+i)\\ \Psi_2^{(i)}(z)&=\frac{(z-z_i)}{(z_{i+1}-z_i)}(1+i)\end{aligned} \tag{24}$$

where $i=1,2,3,\ldots,N$ and z_i and z_{i+1} are, respectively, the lower and upper coordinates of the element i.



Series	Pr	β_e	β_i	Er	δ	t
I	0.71	1.00	0.00	0.01	0.10	0.001
II	7.00	1.00	0.00	0.01	0.10	0.001
III	0.71	1.50	0.00	0.01	0.10	0.001
IV	0.71	1.00	0.40	0.01	0.10	0.001
V	0.71	1.00	0.00	0.05	0.10	0.001
VI	0.71	1.00	0.00	0.01	0.50	0.001
VII	0.71	1.00	0.00	0.01	0.10	0.002

Figure 5. Transient secondary velocity profiles (v) at $Gr=-10$.

Using Equations (23) and (24), Equation (22) reduces to

$$[\bar{A}]\left\{\frac{\partial \mathrm{d}}{\partial t}\right\}+[\bar{B}]\{\mathrm{d}\}=\{\bar{P}\} \tag{25}$$

where

$$\begin{aligned}
\bar{A}_{ij} &= \int_\Omega \Psi_i \Psi_j \, \mathrm{d}z \\
\bar{B}_{ij} &= \int_\Omega \left(-\Psi_i \frac{\partial \Psi_j}{\partial z} + \frac{\partial \Psi_i}{\partial z}\frac{\partial \Psi_j}{\partial z} + M_1 \Psi_i \Psi_j\right) \mathrm{d}z \\
\bar{P}_i &= \int_\Omega \Psi_i \left(Gr\,\theta + M_1 U + \frac{\partial U}{\partial t}\right) \mathrm{d}z
\end{aligned} \tag{26}$$

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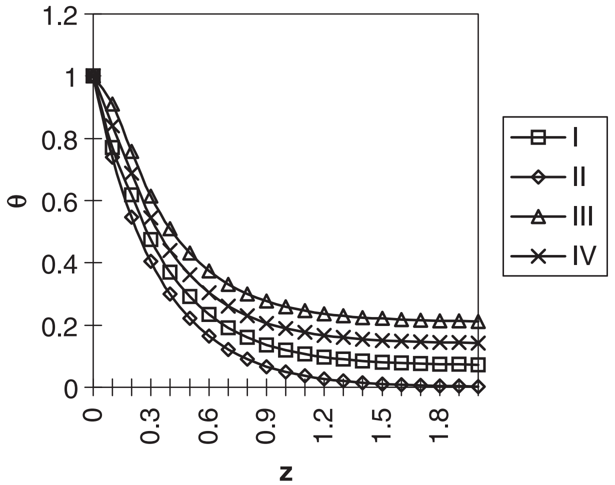

Series	Pr	δ	t
I	0.71	0.10	0.001
II	7.00	0.10	0.001
III	0.71	0.50	0.001
IV	0.71	0.10	0.002

Figure 6. Transient temperature profiles (θ).

Using the Θ-family of operators developed by Reddy [14], Equation (25) takes the form

$$\{\mathrm{d}\}_{n+1} = [\hat{\hat{A}}^{-1}][\hat{\hat{B}}]\{\mathrm{d}\}_n + [\hat{\hat{A}}^{-1}]\{\hat{\hat{P}}\} \tag{27}$$

where

$$\mathrm{d}_0 = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{Bmatrix} \tag{28}$$

The numerical values of the temperature and velocity fields have been computed from Equations (21) and (27) and are shown in Figures 2–6.

3. DISCUSSION OF RESULTS

In order to get a physical insight into the problem and for the purpose of discussing the results, numerical calculations have been carried out for the velocity and temperature profiles and are displayed in Figures 2–6. The numerical method used (the Galerkin finite element method) is unconditionally stable and it is independent of the time step Δt. The figures show the velocity and temperature distributions for constant magnetic field ($M^2 = 5.0$). The velocity profiles are examined for the cases $Gr>0$ and $Gr<0$. $Gr>0$ ($=10$) is used for the case when the flow is in the presence of cooling of the plate by free convection currents. $Gr<0$ ($=-10$) is used for the case when the flow is in the presence of heating of the plate by free convection currents.

From Figures 2 and 4, for the case $Gr = \pm 10$, it is observed that:

i. The primary velocity profile (u) decreases due to an increase in the Hall parameter (β_e) and the ion-slip parameter (β_i).
ii. There is an insignificant change in the primary velocity profile (u) due to an increase in the Prandtl number (Pr), the rotation parameter (Er) and the heat-generating parameter (δ).
iii. The primary velocity profile (u) increases with time (t) for $Gr>0$ (in the presence of cooling of the plate by free convection currents), whereas it decreases with time (t) for $Gr<0$ (in the presence of heating of the plate by free convection currents).

From Figures 3 and 5, for the case $Gr = \pm 10$, it is observed that:

i. The secondary velocity profile (v) decreases due to an increase in the Hall parameter (β_e) and the ion-slip parameter (β_i) whereas it increases due to an increase in the time (t) and the rotation parameter (Er).
ii. There is an insignificant change in the secondary velocity profile (v) due to an increase in the Prandtl number (Pr) and the heat-generating parameter (δ).

From Figure 6, we observed that:

i. The temperature (θ) decreases away from the plate. The decrease is greater for a Newtonian fluid than it is for a non-Newtonian fluid (θ decreases with Pr).
ii The temperature profile (θ) increases due to an increase in the heat-generating parameter (δ) and the time (t).

NOMENCLATURE

σ	electric conductivity
μ_e	magnetic permeability
ω_e	cyclotron frequency
τ_e	electron collision time
e	electric charge
η_e	number density of electron
P_e	electron pressure
$\mathbf{E}$	electric field
$\bar{j}$	electrical current density
$\bar{B}$	magnetic induction

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z	spatial coordinate
t	time coordinate
u	non-dimensional primary velocity
v	non-dimensional secondary velocity
θ	non-dimensional temperature
δ	heat-generating parameter
Pr	Prandtl number
Gr	Grashof number
Er	rotation parameter
M^2	magnetic parameter
β_e, β_i	Hall parameter, ion-slip parameter

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